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THE NUMBER OF t-WISE BALANCED DESIGNS

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We prove that the number of t-wise balanced designs of order n is asymptotically $n^{\left(\binom{n}{t}/(t+1)\right)(1+o(1))}$, provided that blocks of size t are permitted. In the process, we prove that the number of t-profiles (multisets of block sizes) is bounded below by $\exp(c_1 = \sqrt{n}\log n)$ and above by $\exp(c_2\sqrt{n}\log n)$ for constants $c_2 > c_1 > 0$.

1. Preliminaries

A *t-wise balanced design* is a pair $(V, \mathcal{B})$, where V is an n-set of elements, and $\mathcal{B}$ is a set of subsets (called *blocks*) of V, so that every t-subset of V is contained in precisely one set of $\mathcal{B}$. A t-wise balanced design is *proper* if $\mathcal{B}$ contains no blocks of size t; we generally allow blocks of size t. A *t-design* is a proper t-wise balanced design with all blocks of the same cardinality. The *t-profile* of a t-wise balanced design $(V, \mathcal{B})$ is the multiset of the sizes of the blocks in $\mathcal{B}$.

Naturally, when blocks of size t are permitted, there are many t-profiles, and many t-wise balanced designs. We determine the number of t-profiles $P_t(n)$, and the number of t-wise balanced designs $N_t(n)$ asymptotically. More precisely, we prove the following two theorems.

Theorem 1. *There are positive constants c_1, c_2 for which*

$$\exp(c_1\sqrt{n}\log n) \le P_t(n) \le \exp(c_2\sqrt{n}\log n)$$

for fixed $t \ge 2$.

Theorem 2. *The number of distinct (or nonisomorphic) t-wise balanced designs of order n is*

$$N_t(n) = n^{\left[\binom{n}{t}/(t+1)\right](1+o(1))}$$

for fixed $t \ge 2$.

We often use the symbol $o(1)$ to denote a quantity depending on a natural number n, the value of which tends to zero as n tends to infinity. For a function $f(n)$, $o(f(n)) = o(1)f(n)$. For two functions $f(n)$, $g(n)$, we write $f(n) \sim g(n)$ if $f(n) = (1 + o(1))g(n)$. We also write $f(n) \sim_\delta g(n)$ if $|f(n) - g(n)| < \delta g(n)$.

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We prove Theorem 1 in Section 2. In Section 3, we establish the upper bound in Theorem 2, and then in Section 4, we establish the lower bound.

Previous work has concentrated on proper t-designs for $t = 2$ and 3. When $t = 2$, the lower bound in Theorem 2 on 2-wise (pairwise) balanced designs is the same as are best known lower bound for 2-designs with block size 3 (Steiner triple systems); this lower bound was established by Aleksejev [1] and Wilson [7], and later simplified by Phelps [5]. Similarly when $t = 3$, the lower bound in Theorem 2 on 3-wise balanced designs is the same as the best known lower bound for 3-designs with block size 4 (Steiner quadruple systems) [4]. Hence in these cases we find that the number of t-designs with block size $t+1$ is essentially the same as the total number of t-wise balanced designs. However, our Theorem 2 falls far short of determining whether this holds for all t, as our methods rely strongly on the admission of blocks of size t.

2. The Number of t-profiles

In this section, we prove Theorem 1. The case $t = 2$ is settled in the following theorem of Colbourn, Phelps and Rödl [2]:

Theorem A. *There are positive constants c_l and c_u for which*

$$\exp(c_l\sqrt{n}\log n) \leq P_2(n) \leq \exp(c_u\sqrt{n}\log n).$$

The lower bound in Theorem 1 follows immediately from Theorem A. Any pairwise balanced design of order $n - t + 2$ can be extended to a t-wise balanced design by adding $t - 2$ fixed new elements to each block of the pairwise balanced design, and then adding blocks of size t to cover all remaining t-subsets. Hence the number of t-profiles grows at least as quickly as the number of 2-profiles.

Next we establish the upper bound. Let P be a t-profile. Partition the multiset P into two multisets S and L; S contains all block sizes which are at most $10(t-1)\sqrt{n}$, while L contains those larger than that. The number of ways in which S can be chosen is bounded above by $n^{c\sqrt{n}}$, for c a constant. Hence we must only bound the number of ways in which L can be settled. The number p of blocks corresponding to L satisfies

$$10(t-1)p\sqrt{n} - (t-1)\binom{p}{2} \leq n$$

since any two distinct blocks intersect in at most $t - 1$ elements. Moreover, since large blocks can always be replaced by smaller ones, the inequality must hold for all numbers q of blocks, $1 \leq q < p$. Since the inequality does not hold for $\sqrt{n}/5$, we must have $p < \sqrt{n}/5$. Thus there are at most $n^{c'\sqrt{n}}$ choices for L. Combining all choices for S and L gives an upper bound on the number of t-profiles of $\exp(c_2\sqrt{n}\log n)$, for c_2 a constant. This upper bound argument was essentially suggested by Erdős [3].

3. The upper bound in Theorem 2

Let $\mathcal{D} = (V, \mathcal{B})$ be a t-wise balanced design on N elements. Order the blocks of $\mathcal{B}$ as $B_1, \ldots, B_b$ so that

(1) if $i < j$ and $|B_i| > t+4$, $|B_i| \geq |B_j|$, and
(2) if $i < j$ and $|B_i| \leq t+4$, the lexicographically smallest t-subset of B_i precedes the lexicographically smallest t-subset of B_j.

The *ordered t-profile* $\overline{P}$ of $\mathcal{D}$ is then $\langle |B_1|, |B_2|, \dots |B_b| \rangle$. First, we determine an upper bound on the number $N_t^{\overline{P}}(N)$ of t-wise balanced designs whose ordered t-profile is $\overline{P}$. Let $s = \max\{i : |B_i| > t+4\}$. To establish an upper bound on the number of selections for $B_1, \dots, B_s$, observe that there are fewer than $N^{|B_i|}$ ways to select B_i. Once $B_1, \dots, B_r$ $(r \geq s)$ are selected, the next block in the ordering must contain the lexicographically smallest t-subset not yet contained in a block; hence there are fewer than $N^{|B_i|-t}$ ways to select B_i for $i > s$. Combining these observations, we have

$$N_t^{\overline{P}}(N) < \Big(\prod_{i=1}^{s} N^{|B_i|}\Big)\Big(\prod_{i=s+1}^{b} N^{|B_i|-t}\Big). \tag{3.1}$$

For $|B_i| > t+4$, it is easily verified that

$$|B_i| \leq \binom{|B_i|}{t}/(t+1). \tag{3.2}$$

For $|B_i| \leq t+4$, it is easily verified that

$$|B_i| - t \leq \binom{|B_i|}{t}/(t+1), \tag{3.3}$$

with equality when $|B_i| = t+1$ or $t+2$.

From (3.1), (3.2) and (3.3), we have

$$N_t^{\overline{P}}(N) \leq \prod_{i=1}^{b} N^{\binom{|B_i|}{t}/(t+1)}. \tag{3.4}$$

Since $\sum_{i=1}^{b} \binom{|B_i|}{t} = \binom{N}{t}$, (3.4) implies that

$$N_t^{\overline{P}}(N) \leq N^{\binom{N}{t}/(t+1)}. \tag{3.5}$$

Next we bound the number $N_t^P(N)$ of t-wise balanced designs with t-profile P. For a given t-profile P, the number of ordered t-profiles consistent with P is certainly less than $5^{\binom{N}{t}}$, and hence by (3.5),

$$N_t^P(N) \leq N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))}. \tag{3.6}$$

Finally, we bound $N_t(N)$. Let $\mathcal{P}$ be the set of all t-profiles. Then

$$N_t(N) = \sum_{P \in \mathcal{P}} N_t^P(N). \tag{3.7}$$

Using (3.6) to overestimate the maximum number of t-wise balanced designs with given profile, and using Theorem 1 to overestimate the number of t-profiles, we obtain the desired inequality:

$$N_t(N) \leq N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))} \qquad \blacksquare$$

4. The lower bound in Theorem 2

To establish the lower bound, it is convenient to prove a closely related result:

Theorem 3. *Let t, N be positive integers. Then there exist $N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))}$ set systems $\mathscr{S}$ on $\{1,\ldots,N\}$ such that*
(i) $|\mathscr{S}| = t+1$ for every $S \in \mathscr{S}$,
(ii) $|\mathscr{S}| \leq \left[\binom{N}{t}/(t+1)\right](1+o(1))$, and
(iii) there are at most $o(N^t)$ t-subsets of $\{1,\ldots,N\}$ which are not contained in some $S \in \mathscr{S}$.

First we establish the lower bound in Theorem 2 as a consequence of Theorem 3. In every set system $\mathscr{S}$ satisfying properties (i), (ii) and (iii), there are at most $f(N,t) = o(N^t)$ sets $S \in \mathscr{S}$ for which there exists $S' \in \mathscr{S}$ with $|S \cap S'| \geq t$. Deleting all such sets produces a set system $\mathscr{S}^*$ with the property that every t-subset of $\{1,\ldots,N\}$ appears in at most one $S \in \mathscr{S}^*$. Adding all uncovered t-subsets to $\mathscr{S}^*$ as blocks then produces a t-wise balanced design $\mathscr{D}$.

We can obtain the same $\mathscr{S}^*$ (and hence also the same $\mathscr{D}$) from different systems $\mathscr{S}$; however, the number of choices for $\mathscr{S}$ is bounded by $\binom{N}{t}^{f(N,t)}$, and hence in this way we get

$$\frac{N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))}}{\binom{N}{t}^{f(N,t)}} = N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))}$$

different set systems $\mathscr{D}$; each is a t-wise balanced design with blocks of sizes t and $t+1$ only.

Now we prove Theorem 3. The strategy used is quite similar to that used by Rödl [6] in proving the following:

Theorem B. *Let $t < k \ll N$ be positive integers. Then there exists a family $\mathscr{F}$ of k-subsets of N satisfying*
(i) every t-subset of N is contained in at most one member of $\mathscr{F}$, and
(ii) $|\mathscr{F}| \geq \left[\binom{N}{t}\Big/\binom{k}{t}\right](1-o(1))$, where $o(1) \to 0$ as $N \to \infty$.

Although there are a number of similarities with the proof of Theorem B in [6], we include details here whenever it is not possible to refer explicitly to a statement in [6].

In order to prove Theorem 3, we first introduce some notation employed in the proof; subsequently, we establish three lemmas which enable us to prove Theorem 3 at the end of this section.

Let $\mathcal{J}$ be a k-set of positive integers. A *k-partite t-graph* is a pair $G = ((V_j)_{j\in\mathcal{J}}, E)$ such that $|e \cap V_j| \leq 1$ for every $j \in \mathcal{J}$ and $e \in E$, and moreover $e \subset \bigcup_{j\in\mathcal{J}} V_j$ for every $e \in E$. $V(G)$ denotes the vertex set $\bigcup_{j\in\mathcal{J}} V_j$ of G, and $E(G) = E$.

Let $[\mathcal{J}]^t$ denote the set of all t-subsets of $\mathcal{J}$. For $I \in [\mathcal{J}]^t$, $\rho_I = \rho_I(G)$ denotes the cardinality of $E_I(G)$, where

$$E_I(G) = \{e \in E \,:\, e \cap V_i \neq \emptyset \text{ for every } i \in I\}.$$

A subset $R \bigcup_{j\in\mathcal{J}} V_j$, $|R| \geq t$ is *complete* if $[R]^t \subset E$.

From now on, we assume that $k = t+1$. For $R \in E(G)$, $\sigma^R(G)$ denotes the number of complete $(t+1)$-sets containing R. Let $A_1, \ldots, A_p$, $A = \bigcup_{i=1}^{p} A_i$ be pairwise disjoint sets. Then $[\{A_i\}_{i=1}^p]^t$ denotes the system of all t-subsets of A which intersect each A_i in at most one element.

Finally, we require the following auxiliary claim [6]:

Claim: For every pair of positive integers n, m with $n > m$, and positive reals p, q with $p + q = 1$ for which $(2p-1)n < m < 2pn$, we have

$$(*) \qquad \binom{n}{m} p^m q^{n-m} < \exp\Big(-\frac{1}{3}\frac{(m-np)^2}{npq}\Big).$$

The proof of Theorem 3 is divided into three lemmas.

Lemma 4.1. *Let $G = ((V_i)_{i=1}^{t+1}, E)$, $n = |V_1| = |V_2| = \ldots = |V_{t+1}|$ be a $(t+1)$-partite t-graph. Let ρ and σ be positive reals less than one such that*
(i) $\sigma^R(G) \sim \sigma n$ for every edge $\in E$, and
(ii) $\rho_I \sim \rho n^t$ for every $I \in [\{1, 2, \ldots, t+1\}]^t$.

Then for every $\epsilon > 0$, $\delta > 0$, and n sufficiently large, there are $n^{\epsilon\sigma\rho(1-\delta)n^t}$ systems $\mathcal{K}$ of blocks (complete $(t+1)$-sets) from G such that if we put

$$\begin{aligned} G^* &= ((V_i)_{i=1}^{t+1}, E \setminus \{R \,:\, \exists K \in \mathcal{K},\ R \in K\}) \\ \rho_I^* &= \rho_I(G^*) \\ \sigma_R^* &= \sigma^R(G^*) \end{aligned}$$

the following hold:
(a) $\rho_i^* \sim (\rho\exp(-\epsilon\sigma))n^t$ for every $I \in [\{1, 2, \ldots, t+1\}]^t$.
(b) $\sigma_R^* \sim (\sigma\exp(-\epsilon\sigma\rho))n$ for every edge R of G^*.
(c) $|\{\{K_1, K_2\} \,:\, K_1, K_2 \in \mathcal{K},\ K_1 \cap K_2 \neq \emptyset\}| \leq 2\epsilon\sigma|\bigcup_{K\in\mathcal{K}} K|$.

Proof of Lemma 4.1. Let $G = ((V_i)_{i=1}^{t+1}, E)$ be a given $(t+1)$-partite t-graph with properties (i) and (ii) of Lemma 4.1. Suppose without loss of generality that

$n = |V_1| = \ldots = |V_{t+1}|$ is a sufficiently large positive integer (chosen to satisfy constraints which become clear later in the proof). Let Φ be a random variable whose values are subsets of the set $\mathcal{K}(G)$ of all complete $(t+1)$-gons of the graph G. Each $K \in \mathcal{K}(G)$ has $\text{Prob}[K \in \Phi] = \epsilon/n$, and these probabilities are independent for different $K \in \mathcal{K}(G)$.

First we examine edges of G which are not covered by the $(t+1)$-gons chosen in $\mathcal{K}$. To be more exact, observe that $\Gamma = (V(G), E(G) \setminus \{R : \exists K \in \Phi,\ R \in K\})$ is a random variable whose values are subgraphs of G. Now let $I \in [\{1,2,\ldots,t+1\}]^t$. For a fixed edge $R \in E_I(G)$, the probability that R remains in $E_I(\Gamma)$ is

$$p_R = (1 - (\epsilon/n))^{\sigma n(1+o(1))} \sim \exp(-\epsilon\sigma).$$

These probabilities are independent for different $R \in E_I$. The probability that exactly s edges in E_I are not covered by any $(t+1)$-gon $K \in \Phi$ is therefore

$$\sum_{X \in [E_I]^s} \prod_{R \in X} p_R \prod_{R \in E_I \setminus X} (1 - p_R)$$
$$= \binom{\rho n^t(1+o(1))}{s} (\exp(-\epsilon\sigma))^s (1 - \exp(-\epsilon\sigma))^{\rho n^t(1+o(1)) - s} (1 + o(1))^{\rho n^t}.$$

Now let μ be a positive real satisfying

$$\mu \le \min\left(\frac{\delta}{2}, \frac{\epsilon\sigma}{10}\exp(\epsilon\sigma)\right). \tag{4.1}$$

Let S_μ be the set of integers s for which $0 \le s \le \rho n^t(1+o(1)) = |E_I|$, and

$$|s - (\exp(-\epsilon\sigma))\rho n^t| > \mu\rho n^t \exp(-\epsilon\sigma).$$

Then applying $(*)$ gives

$$\sum_{s \in S_\mu} \binom{\rho n^t(1+o(1))}{s} (\exp(-\epsilon\sigma))^s (1 - \exp(-\epsilon\sigma))^{\rho n^t(1+o(1)) - s} (1 + o(1))^{\rho n^t}$$
$$< n^t \exp(-c_1 n^t) < \exp(-c_2 n^t)$$

for some c_1, $c_2 > 0$ and n sufficiently large.

We conclude that

$$\rho_I(\Gamma) \sim_\mu \rho n^t \exp(-\epsilon\sigma) \text{ for every } I \in [\{1,2,\ldots,t+1\}]^t \tag{4.2}$$

with probability larger than $1-(t+1)\exp(-c_2 n^t) > 1-\exp(-c_3 n^t)$, $c_3 > 0$ (again, for n sufficiently large). This verifies that (a) of Lemma 4.1 holds with high probability.

Now we verify (b). More precisely, we shall establish that

$$\text{Prob}\,[\sigma^R(\Gamma) \sim_\mu \sigma^R(G)\exp(-\epsilon\sigma\rho)] > 1 - \exp(-c_4 n)$$

for every $R \in E(G^*)$. Without loss of generality, consider a fixed edge $R \in E(G^*)$ such that $R \in E_I(G^*)$ and $I = \{1,2,\ldots,t\}$. By (i) of the lemma, there are $t(R) = \sigma n(1+o(1))$ vertices $v_1, v_2, \ldots, v_{t(R)} \in V_{t+1}$ such that $\{v_i\} \cup P \in E$

for every $i \in \{1,\ldots,t(R)\}$ and $P \in [R]^{t-1}$. Let A_R be the event that $R \in E(\Gamma)$. For every vertex v_i, $1 \leq i \leq t(R)$, B_i denotes the event that all edges $\{v_i\} \cup P$, $P \in [R]^{t-1}$ remain in $E(\Gamma)$. By i if the lemma, for every i, $1 \leq i \leq t(R)$, the number of complete $(t+1)$-sets L' containing v_i for which $R \notin [L']^t$ and $[L']^t \cap [R \cup \{v_i\}]^t \neq \emptyset$ equals $\sigma tn(1+o(1))$. Deletion of such a $(t+1)$-gon L' corresponds to the situation that B_i fails to occur provided that A_R occurs, and hence

$$\operatorname{Prob}(B_i \mid A_R) \sim (1-(\epsilon/n))^{\sigma tn} \sim \exp(-\epsilon\sigma t).$$

The events $(B_i \mid A_R)$ are independent, so $\sum_i P(B_i \mid A_R) \leq P(A_R)$ for fixed R and different i, $1 \leq i \leq t(R)$, because their complements correspond to deletion of complete $(t+1)$-gons which are independent events. Applying $(*)$ in a similar manner as before), we have

$$\operatorname{Prob}[\sigma^R(\Gamma) \sim_\mu t(R)\exp(-\epsilon\sigma t)] > 1-\exp(-c_4 n) \tag{4.3}$$

where $C_4 > 0$ depends only on n.

Now we verify requirement (c), the requirement on intersections of $(t+1)$-gons. For a system $\mathcal{K}$ of $(t+1)$-gons, $c(\mathcal{K})$ denotes the number of pairs K_1, $K_2 \in \mathcal{K}$ for which $E(K_1) \cap E(K_2) \neq \emptyset$. Then the expectation

$$E(C(\Phi)) = (1+o(1))(t+1)\rho n^t \binom{\sigma n}{2}(\epsilon/n)^2 \leq \frac{(1+o(1))}{2}(t+1)\rho\epsilon^2\sigma^2 n^t$$

and thus

$$\operatorname{Prob}[c(\Phi) \leq (3/4)(t+1)\rho\epsilon^2\sigma^2 n^t] \geq (1/3)(1+o(1)).$$

According to (4.2), we have that

$$\left|\bigcup_{K\in\mathcal{K}} K\right| \geq (t+1)\rho n^t(1+o(1)-(1+\mu)\exp(-\epsilon\sigma))$$

holds with probability bigger than $1-\exp(-c_3 n^t)$ for n sufficiently large with respect to μ.

By (4.1), $\mu \leq \epsilon\sigma\exp(\epsilon\sigma)/10$. Thus, for n sufficiently large,

$$\frac{c(\Phi)}{\left|\bigcup_{K\in\Phi} K\right|} \leq \frac{(3/4)(t+1)\rho\epsilon^2\sigma^2 n^t}{(t+1)\rho n^t(1+o(1)-\exp(-\epsilon\sigma)-(\epsilon\sigma/10))}$$

$$\leq \frac{(3/4)\epsilon^2\sigma^2}{o(1)+\epsilon\sigma-(\epsilon^2\sigma^2)/2-(\epsilon\sigma)/10} \tag{4.4}$$

with probability at least $(1/3)(1+o(1))$.

Finally, observe that it follows directly from (i) and (ii) of the lemma that G contains $\sigma\rho n^{t+1}(1+o(1))(t+1)$-gons; hence, using $(*)$,

$$|\Phi| \sim_\mu \epsilon\sigma\rho n^t \tag{4.5}$$

with probability $1-\exp(-c_4 n)$.

Combining (4.2), (4.3), (4.4) and (4.5),

$$(4.6)\qquad K \in \Phi \text{ satisfies (a), (b), (c) with probability at least } (1/3)(1+o(1)).$$

For a system $\mathcal{K}$ of $(t+1)$-gons with $|\mathcal{K}| \sim \epsilon\sigma\rho n^t$,

$$\begin{aligned}\text{Prob}\,[\Phi = \mathcal{K}] &= (\epsilon/n)^{|\mathcal{K}|}(1-(\epsilon/n))^{\sigma\rho n^t(1+o(1))-|\mathcal{K}|}\\ &\le (\epsilon/n)^{(\epsilon/n)\sigma\rho(1-\mu)n^{t+1}}(1-\epsilon/n))^{(1-(\epsilon/n))\epsilon\sigma\rho n^{t+1}}\\ &\sim (1/n)^{\epsilon\sigma\rho(1-\mu)n^t(1+o(1))}.\end{aligned}$$

Using (4.6), there are therefore

$$n^{\epsilon\sigma\rho(1-\mu)n^t(1+o(1))}$$

systems satisfying requirements (a), (b) and (c) of the lemma.

Since $\mu \le \delta/2$ by (4.1), for n sufficiently large, we obtain the required number

$$n^{\epsilon\sigma\rho(1-\delta)n^t}$$

of systems. ∎

Lemma 4.2. *Let $G = ((V_i)_{i=1}^{t+1}, E)$ be a $(t+1)$-partite t-graph satisfying the assumptions of Lemma 4.1, and for which*

$$(4.7)\qquad \rho < \frac{1}{4}$$

holds. Then for any $\delta > 0$, there exist $n^{\rho(1-\delta)n^t}$ systems $\mathcal{S}$ of $(t+1)$-gons of G such that

(a) $|\bigcup_{S\in\mathcal{S}} S| = |E|(1-o(1))$.

(b) $|\mathcal{S}| \le |E|/(t+1)(1+o(1))$.

Proof of Lemma 4.2. Let ν be a given real satisfying

$$(4.8)\qquad \left(\frac{\nu}{2}\right)^{\rho} \le 1/2$$

We will show that, provided $n = |V_1| = \ldots = |V_{t+1}|$ is sufficiently large, there exist

$$(4.9)\qquad n^{\rho n^t(1-\delta)}$$

systems $\mathcal{S}$ of $(t+1)$-gons of G, which together contain all but $\frac{\nu}{2}|E|$ edges, and such that

$$(4.10)\qquad |\mathcal{S}| \le \frac{|E|(1+\frac{\nu}{2})}{t+1}.$$

We construct the systems $\mathcal{S}$ inductively. Each $\mathcal{S}$ is constructed by forming a sequence $G_0, G_1, \ldots, G_\ell$ of t-graphs such that $E(G_\ell) \subset E(G_{\ell-1}) \subset \ldots \subset E(G_0)$; ℓ

will be specified later. Set $G_0 = G$ and $\epsilon = \frac{\nu}{4\sigma(t+1)}$. Lemma 4.1 then ensures that there exist $n^{\epsilon\sigma\rho(1-\delta)n^t}$ systems $\mathcal{K}_0 = \mathcal{K}$ of $(t+1)$-gons satisfying the conditions of Lemma 4.1. Set $G_1 = G$ minus the edges of $\mathcal{K}_0$; we are now in a position to repeat the application of Lemma 4.1.

Suppose that after $j-1$ steps, we have constructed

$$\prod_{i=0}^{j-1} n^{\epsilon\sigma\rho(1-\delta)n^t \exp(-\epsilon\sigma\rho i)} \tag{4.11}$$

t-graphs $G_j = ((V_i)_{i=1}^{t+1}, E_j)$, $E_j \subset E$, and systems $\mathcal{S}_j$ of $(t+1)$-gons covering edges of $E \setminus E_j$ so that

(a) $\sigma^R(G_j) \sim \sigma n \exp(-\epsilon\sigma j t)$ for every $R \in E(G_j)$.

(b) $\rho_I(G_j) \sim \rho n^t \exp(-\epsilon\sigma j)$ for every $I \in [\{1, 2, \ldots, t+1\}]^t$.

(c) $|\mathcal{S}| \le (1 + \frac{\nu}{2})\frac{|E \setminus E_j|}{t+1}$.

Applying Lemma 4.1 to each such G_j, with ϵ set to $\epsilon_j = \epsilon \exp(\epsilon\sigma j t)$, we can select a system $\mathcal{K}_j$ of $(t+1)$-gons in

$$n^{\epsilon\sigma\rho(1-\delta)n^t \exp(-\epsilon\sigma\rho j)} \tag{4.12}$$

ways, so that if we set $G_{j+1} = G_j$ minus the edges in $(t+1)$-gons of $\mathcal{K}_j$, we have:

(a') $\sigma^R(G_{j+1}) \sim \sigma n \exp(-\epsilon\sigma(j+1)t)$ for every edge $R \in E(G_{j+1})$.

(b') $\rho_I(G_{j+1}) \sim \rho n^t \exp(\epsilon\sigma(j+1))$ for every $I \in [\{1, 2, \ldots, t+1\}]^t$.

Choosing at least one $(t+1)$-gon from each pair K_1, K_2 with $K_1 \cap K_2 \ne \emptyset$, and deleting all edges in chosen $(t+1)$-gons leaves a system with at least

$$|\mathcal{K}_j| - 2\epsilon\sigma\Big|\bigcup_{K \in \mathcal{K}_j} K\Big|$$

pairwise disjoint $(t+1)$-gons covering at most $|\bigcup_{K \in \mathcal{K}} K|$ edges. Thus we get

$$|\mathcal{K}_j| \le \frac{\Big|\bigcup_{K \in \mathcal{K}_j} K\Big|}{t+1} + 2\epsilon\sigma\Big|\bigcup_{K \in \mathcal{K}_j} K\Big|.$$

Set $\mathcal{S}_{j+1} = \mathcal{S}_j \cup \mathcal{K}_j$. $\mathcal{S}_{j+1}$ covers all edges of $E \setminus E_{j+1}$, and hence we get

$$\text{(c')} \qquad \begin{aligned} |\mathcal{S}_{j+1}| &= |\mathcal{S}_j| + |\mathcal{K}_j| \\ &\le (1 + \frac{\nu}{2})\frac{|E \setminus E_j|}{t+1} + \frac{1}{t+1}(1 + 2\epsilon\sigma(t+1))\Big|\bigcup_{K \in \mathcal{K}_j} K\Big| \\ &\le (1 + \frac{\nu}{2})\frac{|E \setminus E_j|}{t+1} + \frac{1}{t+1}(1 + \frac{\nu}{2})|E_j \setminus E_{j+1}| \\ &\le \frac{|E \setminus E_{j+1}|}{t+1}. \end{aligned}$$

Finally, combining (4.11) and (4.12), we obtain

$$\prod_{i=0}^{j} n^{\epsilon\sigma\rho(1-\delta)n^t \exp(-\epsilon\sigma\rho i)}$$

ways to select $G_1, \ldots, G_{j+1}$.

Set $\ell = \lceil (1/\epsilon\sigma)\ln(2/\nu)\rceil$, and repeat this procedure ℓ times. It is important to observe that ℓ is not a function of n. Hence, although each application of Lemma 4.1 introduces a small inaccuracy, each such error is a factor of $o(1)$, and since the number of applications of the Lemma is independent of n, the total error introduced is also $o(1)$.

The systems $\mathcal{S} = \mathcal{S}_\ell$ cover all edges of $E \setminus E_\ell$; since $|E_\ell| \leq \exp(-\epsilon\sigma\ell)|E| \leq \frac{\nu}{2}|E|$, (4.9) holds. In addition,

$$|\mathcal{S}_\ell| \leq (1+\frac{\nu}{2})\frac{|E\setminus E_\ell|}{t+1} \leq (1+\frac{\nu}{2})\frac{|E|}{t+1}$$

and thus (4.10) holds as well.

Finally, there are

$$\prod_{i=0}^{\ell-1} n^{\epsilon\sigma\rho(1-\delta)n^t \exp(-\epsilon\sigma\rho i)} \tag{4.13}$$

ways to select $G_1, \ldots, G_\ell$. Now

$$\sum_{i=0}^{\ell-1} \exp(-\epsilon\sigma\rho i) = \frac{1-\exp(-\epsilon\sigma\rho\ell)}{1-\exp(-\epsilon\sigma\rho)}. \tag{4.14}$$

We show now that the right hand side of (4.14) is at least $1/(\epsilon\sigma)$. By (4.7), $\rho < 1/4$, and hence

$$1-\exp(-\epsilon\sigma\rho) \leq \epsilon\sigma/2. \tag{4.15}$$

On the other hand, by (4.8) we have that

$$\exp(-\epsilon\sigma\rho\ell) < 1/2. \tag{4.16}$$

Combining (4.15) and (4.16) shows that the right hand side is indeed at least $1/(\epsilon\sigma)$; by (4.13), we therefore have

$$n^{\rho(1-\delta)n^t} \tag{4.17}$$

ways to select $G_1, \ldots, G_\ell$.

Now we estimate the number of different $\mathcal{S}_\ell$'s which we get by the method described. The same $\mathcal{S}_\ell$ can arise from different choices of $G_1, \ldots, G_\ell$; however, there are at most ℓ^{n^t} such choices, and hence (4.17) implies that there are

$$n^{\rho(1-\delta)n^t}$$

different $\mathcal{S}_\ell$'s as well. ∎

The following was proved in [6]:

Theorem 4.3. *Let $p > t$ be given positive integers. Let $A_1, \ldots, A_p$ be pairwise disjoint sets of the same large cardinality n. Then there exists a decomposition*

$$[\{A_i\}_{i=1}^p]^t = \bigcup\{E_J : J \in [\{1,2,\ldots,p\}]^{t+1}\}$$

such that for every J, $J' \in [\{1,2,\ldots,p\}]^{t+1}$, $J \neq J'$,
(a) $E_J \cap E_{J'} = \emptyset$.
(b) $E_J \subset [\{A_i\}_{i\in J}]^t$, *and the t-graph $H(J)$ defined by $V(H(J)) = \bigcup_{i\in J} A_i$, $E(H(J)) = E_J$ satisfies*
(c) $\rho_I(H(J)) \sim (1/u)n^t$ *for every $I \in [J]^t$,*
(d) $\sigma^R(H(J)) \sim \sigma n$ *for every $R \in E_j$,*
where $\sigma = (1/u)^t$ and $u = p - t$. ∎

Proof of Theorem 3. Take two large positive integers, p, n, $p \ll n$. Set $N = np$. Consider p pairwise disjoint sets $A_1, A_2, \ldots, A_p$ of the same cardinality n. Lemma 4.3 ensures the existence of a decomposition

$$[\{A_i\}_{i=1}^p]^t = \bigcup\{E_J : J \in [\{1,2,\ldots,p\}]^{t+1}\}.$$

Thus for each $J \in [\{1,2,\ldots,p\}]^{t+1}$, we have the t-graph $H(J) = ((A_j)_{j\in J}, E_J)$, which satisfies the assumptions of Lemma 4.2. Hence, for n sufficiently large, we get

$$n^{(1/u)n^t(1-o(1))} \tag{4.18}$$

system $\mathcal{S}(J)$ of $(t+1)$-gons covering almost all edges in $[\{A_j\}_{j\in J}]^t$ precisely once (i.e., satisfying Lemma 4.2).

Taking the disjoint union of the systems $\mathcal{S}(J)$ over all $J \in [\{1,2,\ldots,p\}]^{t+1}$, we get systems which satisfy the requirements of Theorem 3. This follows from the fact that for p large enough, almost all t-subsets of $\bigcup_{i=1}^{p} A_i$ are elements of $[\{A_i\}_{i=1}^p]^t$. By (4.18), we construct

$$n^{(1/u)n^t\binom{p}{t+1}} \sim N^{\left[\binom{N}{t}/(t+1)\right](1+o(1))}$$

different systems. This completes the proof of Theorem 3, and hence also Theorem 2. ∎

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